

GORENSTEIN FANO POLYTOPES ARISING FROM ORDER POLYTOPES AND CHAIN POLYTOPES

TAKAYUKI HIBI, KAZUNORI MATSUDA, AND AKIYOSHI TSUCHIYA

ABSTRACT. Richard Stanley introduced the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ arising from a finite partially ordered set P , and showed that the Ehrhart polynomial of $\mathcal{O}(P)$ is equal to that of $\mathcal{C}(P)$. In addition, the unimodular equivalence problem of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ was studied by the first author and Nan Li. In the present paper, three integral convex polytopes $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$, where P and Q are partially ordered sets with $|P| = |Q|$, will be studied. First, it will be shown that the Ehrhart polynomial of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ coincides with that of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$. Furthermore, when P and Q possess a common linear extension, it will be proved that these three convex polytopes have the same Ehrhart polynomial. Second, the problem of characterizing partially ordered sets P and Q for which $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ or $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ or $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is a smooth Fano polytope will be solved. Finally, when these three polytopes are smooth Fano polytopes, the unimodular equivalence problem of these three polytopes will be discussed.

INTRODUCTION

A convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is called *integral* if all of vertices of \mathcal{P} belong to \mathbb{Z}^d . Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d . Given integers $n = 1, 2, \dots$, we define the function $i(\mathcal{P}, n)$ as follows:

$$i(\mathcal{P}, n) := |(n\mathcal{P} \cap \mathbb{Z}^d)|,$$

where $n\mathcal{P} = \{n\alpha \mid \alpha \in \mathcal{P}\}$. We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of \mathcal{P} . It is known that $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$ (see [3]).

Next, we introduce some classes of Fano polytopes. Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d .

- We say that \mathcal{P} is a *Fano polytope* if the origin of \mathbb{R}^d is the unique integer point belonging to the interior of \mathcal{P} .
- A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope \mathcal{P}^\vee of a Fano polytope \mathcal{P} is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of \mathbb{R}^d .)

2010 *Mathematics Subject Classification.* 13P10, 52B20.

Key words and phrases. order polytope, chain polytope, Ehrhart polynomial, smooth Fano polytope, unimodular equivalence.

- A *\mathbb{Q} -factorial Fano polytope* is a simplicial Fano polytope, i.e., a Fano polytope each of whose faces is a simplex.
- A *smooth Fano polytope* is a Fano polytope such that the vertices of each facet form a \mathbb{Z} -basis of \mathbb{Z}^d .

Thus in particular a smooth Fano polytope is \mathbb{Q} -factorial and Gorenstein.

We recall some terminologies of partially ordered sets. Let $P = \{p_1, \dots, p_d\}$ be a partially ordered set. A *linear extension* of P is a permutation $\sigma = i_1 i_2 \dots i_d$ of $[d] = \{1, 2, \dots, d\}$ which satisfies $i_a < i_b$ if $p_{i_a} < p_{i_b}$ in P . A subset I of P is called a *poset ideal* of P if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Note that the empty set \emptyset and P itself are poset ideals of P . Let $\mathcal{J}(P)$ denote the set of poset ideals of P . A subset A of P is called an *antichain* of P if p_i and p_j belonging to A with $i \neq j$ are incomparable. In particular, the empty set \emptyset and each 1-element subsets $\{p_j\}$ are antichains of P . Let $\mathcal{A}(P)$ denote the set of antichains of P . For each subset $I \subset P$, we define the $(0, 1)$ -vectors $\rho(I) = \sum_{p_i \in I} \mathbf{e}_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the canonical unit coordinate vectors of \mathbb{R}^d . In particular $\rho(\emptyset)$ is the origin $\mathbf{0}$ of \mathbb{R}^d .

In [15], Stanley introduced the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ arising from a partially ordered set P . It is known that both $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are d -dimensional convex polytopes, and

$$\{\text{the sets of vertices of } \mathcal{O}(P)\} = \{\rho(I) \mid I \in \mathcal{J}(P)\},$$

$$\{\text{the sets of vertices of } \mathcal{C}(P)\} = \{\rho(A) \mid A \in \mathcal{A}(P)\}$$

follows ([15, Corollary 1.3, Theorem 2.2]). Moreover, $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same Ehrhart polynomial ([15, Theorem 4.1]). In particular, the volume of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of P ([15, Corollary 4.2]).

In present papers, as analogies of the order polytope and the chain polytope, the integral convex polytopes associated with two partially ordered sets are studied. These polytopes are given by combining the order polytopes and the chain polytopes.

Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite partially ordered sets with $|P| = |Q| = d$. We define integer matrices $\Psi(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Psi(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Psi(\mathcal{C}(P), -\mathcal{C}(Q))$ as follows:

$$\begin{aligned} \Psi(\mathcal{O}(P), -\mathcal{O}(Q)) &= \{\rho(I) \mid \emptyset \neq I \in \mathcal{J}(P)\} \cup \{-\rho(J) \mid \emptyset \neq J \in \mathcal{J}(Q)\} \cup \{\mathbf{0}\}, \\ \Psi(\mathcal{O}(P), -\mathcal{C}(Q)) &= \{\rho(I) \mid \emptyset \neq I \in \mathcal{J}(P)\} \cup \{-\rho(J) \mid \emptyset \neq J \in \mathcal{A}(Q)\} \cup \{\mathbf{0}\}, \\ \Psi(\mathcal{C}(P), -\mathcal{C}(Q)) &= \{\rho(I) \mid \emptyset \neq I \in \mathcal{A}(P)\} \cup \{-\rho(J) \mid \emptyset \neq J \in \mathcal{A}(Q)\} \cup \{\mathbf{0}\} \end{aligned}$$

and we write $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q)) \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\Psi(\mathcal{O}(P), -\mathcal{C}(Q))$. Similarly, we define $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ as the convex hull of $\Psi(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Psi(\mathcal{C}(P), -\mathcal{C}(Q))$, respectively. These

polytopes are analogies of the order polytope and the chain polytope, and are generalizations of the convex polytope arising from the centrally symmetric configuration (see [13]).

We note that these are d -dimensional polytopes. Moreover, since $\rho(P) = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathcal{O}(P)$ and $\rho(\{q_j\}) = \mathbf{e}_j \in \mathcal{C}(Q)$ for $1 \leq j \leq d$, we have that the origin $\mathbf{0}$ of \mathbb{R}^d is belonging to the interior of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and that of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$. However, it is not necessarily that $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ has the same property. It is known that the origin $\mathbf{0}$ of \mathbb{R}^d is belonging to the interior of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ if and only if P and Q possess a common linear extension ([9, Lemma 1.1]). In addition, it is also known that $\Gamma(\mathcal{O}(P), -\mathcal{O}(P))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are always Gorenstein Fano ([10, Corollary 2.3], [11, Corollary 1.2], [14, Theorem 2.8]) and $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is Gorenstein Fano if and only if P and Q possess a common linear extension ([9, Corollary 2.2]). Hence to determine when these polytopes are smooth Fano is an important problem. Similarly, the question whether these polytopes are unimodularly equivalent when these polytopes are smooth is also interesting. The problem when $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodularly equivalent was solved in [8].

This paper is organized as follows. In Section 1, we study the Ehrhart polynomials of these polytopes (Theorem 1.1). This is an analogy of Stanley's results mentioned before. In Section 2, we study the characterization problem of partially ordered sets yield smooth Fano polytopes (Theorems 2.1, 2.2 and 2.3). Finally, in Section 3, we study the unimodular equivalence of smooth Fano polytopes. In fact, we show $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent, however, these polytopes are not unimodularly equivalent to $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$, when all polytopes are smooth (Theorem 3.1).

For fundamental materials on Gröbner bases and toric ideals, see [6].

1. SQUAREFREE QUADRATIC GRÖBNER BASIS AND EHRHART POLYNOMIAL

In this section, we show the following:

Theorem 1.1. *Work with the same notation as in Introduction. Then we have*

$$i(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)), n) = i(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)), n).$$

In particular, the volume of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is the same as that of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$. Moreover, if P and Q possess a common linear extension, then we have

$$i(\Gamma(\mathcal{O}(P), -\mathcal{O}(Q)), n) = i(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)), n) = i(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)), n).$$

In this case, these polytopes have the same volume.

In order to prove this theorem, we use the following facts. We say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is *normal* if, for each integer $N > 0$ and for each $\mathbf{a} \in N\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathcal{P} \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_N$.

- Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of $\dim \mathcal{P} = d$ and

$$K[\mathcal{P}] := K[x_1^{\alpha_1} \cdots x_d^{\alpha_d} t \mid (\alpha_1, \dots, \alpha_d) \in \mathcal{P}] \subset K[x_1, \dots, x_d, t]$$

be the toric ring of \mathcal{P} over a field K . Assume that there exists a monomial order $<$ on $K[x_1, \dots, x_d, t]$ such that the initial ideal $\text{in}_<(I_{\mathcal{P}})$ of the toric ideal of $K[\mathcal{P}]$ with respect to the order $<$ is squarefree. Then \mathcal{P} is a normal polytope. (see [6])

- Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope. If \mathcal{P} is normal, then the Ehrhart polynomial of \mathcal{P} is equal to the Hilbert polynomial of the toric ring $K[\mathcal{P}]$.
- Let S be a polynomial ring and $I \subset S$ be a graded ideal of S . Let $<$ be a monomial order on S . Then S/I and $S/\text{in}_<(I)$ have the same Hilbert function. (see [4, Corollary 6.1.5])

At first, we define the toric rings $K[\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))]$ (resp. $K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ and $K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))]$) of the polytope $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ (resp. $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$), and prove the normality of these polytopes by using the theory of toric ideals.

Let, as before, $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite partially ordered sets with $|P| = |Q| = d$. For a poset ideal of P or Q , we write $\max(I)$ for the set of maximal elements of I . In particular, $\max(I)$ is an antichain. Note that for each antichain A , there exists a poset ideal I such that $A = \max(I)$.

Let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}, s]$ denote the Laurent polynomial ring in $2d + 1$ variables over a field K . If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, then $\mathbf{t}^\alpha s$ is the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$. In particular $\mathbf{t}^0 s = s$. Then we define the toric rings $K[\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))]$, $K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ and $K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))]$ as follows:

$$K[\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))] = K[\mathbf{t}^\alpha s \mid \alpha \in \Gamma(\mathcal{O}(P), -\mathcal{O}(Q))],$$

$$K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))] = K[\mathbf{t}^\alpha s \mid \alpha \in \Gamma(\mathcal{O}(P), -\mathcal{C}(Q))],$$

$$K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))] = K[\mathbf{t}^\alpha s \mid \alpha \in \Gamma(\mathcal{C}(P), -\mathcal{C}(Q))].$$

Let

$$\begin{aligned} K[\mathcal{OO}] &= K[\{x_I\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_J\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}], \\ K[\mathcal{OC}] &= K[\{x_I\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}], \\ K[\mathcal{CC}] &= K[\{x_{\max(I)}\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}] \end{aligned}$$

denote the polynomial rings over K , and define the surjective ring homomorphisms $\pi_{\mathcal{OO}}$, $\pi_{\mathcal{OC}}$ and $\pi_{\mathcal{CC}}$ by the following:

- $\pi_{\mathcal{OO}} : K[\mathcal{OO}] \rightarrow K[\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))]$ by setting
 $\pi_{\mathcal{OO}}(x_I) = \mathbf{t}^{\rho(I)}s$, $\pi_{\mathcal{OO}}(y_J) = \mathbf{t}^{-\rho(J)}s$ and $\pi_{\mathcal{OO}}(z) = s$,
- $\pi_{\mathcal{OC}} : K[\mathcal{OC}] \rightarrow K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ by setting
 $\pi_{\mathcal{OC}}(x_I) = \mathbf{t}^{\rho(I)}s$, $\pi_{\mathcal{OC}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J))}s$ and $\pi_{\mathcal{OC}}(z) = s$,
- $\pi_{\mathcal{CC}} : K[\mathcal{CC}] \rightarrow K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))]$ by setting
 $\pi_{\mathcal{CC}}(x_{\max(I)}) = \mathbf{t}^{\rho(\max(I))}s$, $\pi_{\mathcal{CC}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J))}s$ and $\pi_{\mathcal{CC}}(z) = s$

where $\emptyset \neq I \in \mathcal{J}(P)$ and $\emptyset \neq J \in \mathcal{J}(Q)$. Then the *toric ideal* $I_{\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))}$ of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is the kernel of $\pi_{\mathcal{OO}}$. Similarly, the toric ideal $I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))}$ (resp. $I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$) is the kernel of $\pi_{\mathcal{OC}}$ (resp. $\pi_{\mathcal{CC}}$).

Next, we introduce monomial orders $<_{\mathcal{OO}}$, $<_{\mathcal{OC}}$ and $<_{\mathcal{CC}}$ and $\mathcal{G}_{\mathcal{OO}}$, $\mathcal{G}_{\mathcal{OC}}$ and $\mathcal{G}_{\mathcal{CC}}$ which are the set of binomials.

Let $<_{\mathcal{OO}}$ denote a reverse lexicographic order on $K[\mathcal{OO}]$ satisfying

- $z <_{\mathcal{OO}} y_J <_{\mathcal{OO}} x_I$;
- $x_{I'} <_{\mathcal{OO}} x_I$ if $I' \subset I$;
- $y_{J'} <_{\mathcal{OO}} y_J$ if $J' \subset J$,

and $\mathcal{G}_{\mathcal{OO}}$ the set of the following binomials:

- (i) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;
- (ii) $y_J y_{J'} - y_{J \cup J'} y_{J \cap J'}$;
- (iii) $x_I y_J - x_{I \setminus \{p_i\}} y_{J \setminus \{q_i\}}$,

and let $<_{\mathcal{OC}}$ denote a reverse lexicographic order on $K[\mathcal{OC}]$ satisfying

- $z <_{\mathcal{OC}} y_{\max(J)} <_{\mathcal{OC}} x_I$;
- $x_{I'} <_{\mathcal{OC}} x_I$ if $I' \subset I$;
- $y_{\max(J')} <_{\mathcal{OC}} y_{\max(J)}$ if $J' \subset J$,

and $\mathcal{G}_{\mathcal{OC}}$ the set of the following binomials:

- (iv) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;
- (v) $y_{\max(J)} y_{\max(J')} - y_{\max(J \cup J')} y_{\max(J * J')}$;
- (vi) $x_I y_{\max(J)} - x_{I \setminus \{p_i\}} y_{\max(J) \setminus \{q_i\}}$,

and let $<_{\mathcal{CC}}$ denote a reverse lexicographic order on $K[\mathcal{CC}]$ satisfying

- $z <_{\mathcal{CC}} y_{\max(J)} <_{\mathcal{CC}} x_{\max(I)}$;
- $x_{\max(I')} <_{\mathcal{CC}} x_{\max(I)}$ if $I' \subset I$;
- $y_{\max(J')} <_{\mathcal{CC}} y_{\max(J)}$ if $J' \subset J$,

and $\mathcal{G}_{\mathcal{CC}}$ the set of the following binomials:

- (vii) $x_{\max(I)} x_{\max(I')} - y_{\max(I \cup I')} y_{\max(I * I')}$;
- (viii) $y_{\max(J)} y_{\max(J')} - y_{\max(J \cup J')} y_{\max(J * J')}$;
- (ix) $x_{\max(I)} y_{\max(J)} - x_{\max(I) \setminus \{p_i\}} y_{\max(J) \setminus \{q_i\}}$,

where

- $x_\emptyset = y_\emptyset = z$;
- I and I' are poset ideals of P which are incomparable in $\mathcal{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathcal{J}(Q)$;
- $I * I'$ is the poset ideal of P generated by $\max(I \cap I') \cap (\max(I) \cup \max(I'))$;
- $J * J'$ is the poset ideal of Q generated by $\max(J \cap J') \cap (\max(J) \cup \max(J'))$;
- p_i is a maximal element of I and q_i is a maximal element of J .

It is known that $\mathcal{G}_{\mathcal{O}\mathcal{O}}$ is a Gröbner basis of $I_{\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))}$ with respect to $<_{\mathcal{O}\mathcal{O}}$ ([9, Theorem 2.1]) and $\mathcal{G}_{\mathcal{O}\mathcal{C}}$ is a Gröbner basis of $I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))}$ with respect to $<_{\mathcal{O}\mathcal{C}}$ ([11, Theorem 1.1]). As corollaries, we have the normality of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ ([9, Corollary 2.2], [11, Corollary 1.2]). However, the polytope $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ was studied in [14] in more general situation, the normality of this polytope is still open. To prove that $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is normal, we compute a Gröbner basis of $I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$ with respect to $<_{\mathcal{C}\mathcal{C}}$.

Proposition 1.2. *Work with the same situation as above. Then $\mathcal{G}_{\mathcal{C}\mathcal{C}}$ is a Gröbner basis of $I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$ with respect to $<_{\mathcal{C}\mathcal{C}}$.*

Proof. First, it is clear that $\mathcal{G}_{\mathcal{C}\mathcal{C}} \subset I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$. For a binomial $f = u - v$, we call u the *first* monomial of f and we call v the *second* monomial of f . By the definition of $<_{\mathcal{C}\mathcal{C}}$, the initial monomial of each of the binomials (vii) – (ix) with respect to $<_{\mathcal{C}\mathcal{C}}$ is its first monomial. Let $\text{in}_{<_{\mathcal{C}\mathcal{C}}}(\mathcal{G}_{\mathcal{C}\mathcal{C}})$ denote the set of initial monomials of binomials belonging to $\mathcal{G}_{\mathcal{C}\mathcal{C}}$. From [12, (0.1)], it follows that, in order to show that $\mathcal{G}_{\mathcal{C}\mathcal{C}}$ is a Gröbner basis of $I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$ with respect to $<_{\mathcal{C}\mathcal{C}}$, we need to prove the following:

(♣) If u and v are monomials belonging to $K[\mathcal{C}\mathcal{C}]$ with $u \neq v$ such that $u \notin \langle \text{in}_{<_{\mathcal{C}\mathcal{C}}}(\mathcal{G}_{\mathcal{C}\mathcal{C}}) \rangle$ and $v \notin \langle \text{in}_{<_{\mathcal{C}\mathcal{C}}}(\mathcal{G}_{\mathcal{C}\mathcal{C}}) \rangle$, then $\pi_{\mathcal{C}\mathcal{C}}(u) \neq \pi_{\mathcal{C}\mathcal{C}}(v)$.

Let $u, v \in K[\mathcal{C}\mathcal{C}]$ be monomials with $u \neq v$. Write

$$u = z^\alpha x_{\max(I_1)}^{\xi_1} \cdots x_{\max(I_a)}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b},$$

$$v = z^{\alpha'} x_{\max(I'_1)}^{\xi'_1} \cdots x_{\max(I'_{a'})}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}},$$

where

- $\alpha \geq 0, \alpha' \geq 0$;
- $I_1, \dots, I_a, I'_1, \dots, I'_{a'} \in \mathcal{J}(P) \setminus \{\emptyset\}$;
- $J_1, \dots, J_b, J'_1, \dots, J'_{b'} \in \mathcal{J}(Q) \setminus \{\emptyset\}$;
- $\xi_1, \dots, \xi_a, \nu_1, \dots, \nu_b, \xi'_1, \dots, \xi'_{a'}, \nu'_1, \dots, \nu'_{b'} > 0$,

and where u and v are relatively prime with $u \notin \langle \text{in}_{<_{\mathcal{C}\mathcal{C}}}(\mathcal{G}_{\mathcal{C}\mathcal{C}}) \rangle$ and $v \notin \langle \text{in}_{<_{\mathcal{C}\mathcal{C}}}(\mathcal{G}_{\mathcal{C}\mathcal{C}}) \rangle$. Note that either $\alpha = 0$ or $\alpha' = 0$ since u and v are relatively prime. Hence we may assume that $\alpha' = 0$. Thus

$$u = z^\alpha x_{\max(I_1)}^{\xi_1} \cdots x_{\max(I_a)}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b},$$

$$v = x_{\max(I'_1)}^{\xi'_1} \cdots x_{\max(I'_{a'})}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}}.$$

Since the initial monomial of each of the binomials (vii) – (ix) with respect to $<_{cc}$ does not belong to $\langle \text{in}_{<_{cc}}(\mathcal{G}_{cc}) \rangle$, we have that

- $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_a$ and $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_b$;
- $I'_1 \subsetneq I'_2 \subsetneq \cdots \subsetneq I'_{a'}$ and $J'_1 \subsetneq J'_2 \subsetneq \cdots \subsetneq J'_{b'}$.

Furthermore, by virtue of [5] and [7], it suffices to discuss u and v with $(a, a') \neq (0, 0)$ and $(b, b') \neq (0, 0)$.

Since $I_a \neq I'_{a'}$, we may assume that $I_a \setminus I'_{a'} \neq \emptyset$. Then there exists a maximal element p_{i^*} of I_a with $p_{i^*} \notin I'_{a'}$.

Suppose that $\pi_{cc}(u) = \pi_{cc}(v)$. Then we have

$$\sum_{\substack{I \in \{I_1, \dots, I_a\} \\ p_i \in \max(I)}} \xi_I - \sum_{\substack{J \in \{J_1, \dots, J_b\} \\ q_i \in \max(J)}} \nu_J = \sum_{\substack{I' \in \{I'_1, \dots, I'_{a'}\} \\ p_i \in \max(I')}} \xi'_{I'} - \sum_{\substack{J' \in \{J'_1, \dots, J'_{b'}\} \\ q_i \in \max(J')}} \nu'_{J'}.$$

for all $1 \leq i \leq d$ by comparing the degree of t_i . By assumption, $p_{i^*} \notin I'_{a'}$. This means that $p_{i^*} \notin \max(I'_{c'})$ for all $1 \leq c' \leq a'$. Hence we have

$$\sum_{\substack{I \in \{I_1, \dots, I_a\} \\ p_{i^*} \in \max(I)}} \xi_I - \sum_{\substack{J \in \{J_1, \dots, J_b\} \\ q_{i^*} \in \max(J)}} \nu_J = - \sum_{\substack{J' \in \{J'_1, \dots, J'_{b'}\} \\ q_{i^*} \in \max(J')}} \nu'_{J'} \leq 0.$$

Moreover, since p_{i^*} is a maximal element of I_a , we also have

$$\sum_{\substack{I \in \{I_1, \dots, I_a\} \\ p_{i^*} \in \max(I)}} \xi_I > 0.$$

Hence there exists an integer c with $1 \leq c \leq b$ such that $q_{i^*} \in \max(J_c)$. Therefore we have $x_{\max(I_a)} y_{\max(J_c)} \in \langle \text{in}_{<_{cc}}(\mathcal{G}_{cc}) \rangle$, but this is a contradiction. \square

By this proposition, it is clear that the initial ideal $\text{in}_{<_{cc}}(I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))})$ of the toric ideal $I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}$ with respect to the order $<_{cc}$ is squarefree. Hence we have

Corollary 1.3. $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is a normal Gorenstein Fano polytope for any partially ordered sets P and Q with $|P| = |Q| = d$.

Here, we put

$$\begin{aligned} R_{\mathcal{OO}} &:= K[\mathcal{OO}] / \text{in}_{<_{\mathcal{OO}}}(I_{\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))}), \\ R_{\mathcal{OC}} &:= K[\mathcal{OC}] / \text{in}_{<_{\mathcal{OC}}}(I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))}), \\ R_{\mathcal{CC}} &:= K[\mathcal{CC}] / \text{in}_{<_{\mathcal{CC}}}(I_{\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))}). \end{aligned}$$

Next, we prove the following.

Proposition 1.4. The ring $R_{\mathcal{OC}}$ is isomorphic to the ring $R_{\mathcal{CC}}$ for any partially ordered sets P and Q with $|P| = |Q| = d$. Moreover, if P and Q possess a common linear extension, then these rings $R_{\mathcal{OO}}$, $R_{\mathcal{OC}}$ and $R_{\mathcal{CC}}$ are isomorphic.

Proof. From [9, Theorem 2.1], [11, Theorem 1.1] and Proposition 1.2, we have

$$\begin{aligned}
R_{\mathcal{O}\mathcal{O}} &\cong \frac{K[\{x_I\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_J\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}]}{(x_I x_{I'}, y_J y_{J'}, x_I y_J \mid I, I', J \text{ and } J' \text{ satisfy } (*))}, \\
R_{\mathcal{O}\mathcal{C}} &\cong \frac{K[\{x_I\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}]}{(x_I x_{I'}, y_{\max(J)} y_{\max(J')}, x_I y_{\max(J)} \mid I, I', J \text{ and } J' \text{ satisfy } (*))}, \\
R_{\mathcal{C}\mathcal{C}} &\cong \frac{K[\{x_{\max(I)}\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}]}{(x_{\max(I)} x_{\max(I')}, y_{\max(J)} y_{\max(J')}, x_{\max(I)} y_{\max(J)} \mid I, I', J \text{ and } J' \text{ satisfy } (*))},
\end{aligned}$$

where the condition $(*)$ is the following:

- I and I' are poset ideals of P which are incomparable in $\mathcal{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathcal{J}(Q)$;
- There exists $1 \leq i \leq d$ such that p_i is a maximal element of I and q_i is a maximal element of J .

Hence it is easy to see that the ring homomorphism $\varphi : R_{\mathcal{O}\mathcal{C}} \rightarrow R_{\mathcal{C}\mathcal{C}}$ by setting $\varphi(x_I) = x_{\max(I)}$, $\varphi(y_{\max(J)}) = y_{\max(J)}$ and $\varphi(z) = z$ is an isomorphism. Similarly, if P and Q possess a common linear extension, we can see that the ring homomorphism $\varphi' : R_{\mathcal{O}\mathcal{O}} \rightarrow R_{\mathcal{O}\mathcal{C}}$ by setting $\varphi'(x_I) = x_I$, $\varphi'(y_J) = y_{\max(J)}$ and $\varphi'(z) = z$ is an isomorphism. Hence we have $R_{\mathcal{O}\mathcal{O}} \cong R_{\mathcal{O}\mathcal{C}} \cong R_{\mathcal{C}\mathcal{C}}$. \square

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. From [11, Corollary 1.2] and Proposition 1.2, we have that both $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are normal. Hence the Ehrhart polynomial of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ (resp. $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$) is equal to the Hilbert polynomial of $K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ (resp. $K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))]$). By Proposition 1.4, $R_{\mathcal{O}\mathcal{C}}$ and $R_{\mathcal{C}\mathcal{C}}$ have the same Hilbert polynomial. Hence $K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ and $K[\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))]$ also have the same Hilbert polynomial. Therefore we have the desired conclusion.

If P and Q possess a common linear extension, $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is also normal from [9, Corollary 2.2]. Therefore, by the same argument, we have the desired conclusion. \square

We immediately obtain the following corollary.

Corollary 1.5. *For any partially ordered sets P and Q with $|P| = |Q| = d$, we have*

$$i(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)), n) = i(\Gamma(\mathcal{C}(Q), -\mathcal{C}(P)), n).$$

In particular, these polytopes have the same volume.

As the end of this section, we give an example that P and Q do not have any common linear extension.

Example 1.6. Let $P = \{p_1 < p_2\}$ and $Q = \{q_2 < q_1\}$ be chains. It is clear that P and Q have no common linear extension. Then

$$i(\Gamma(\mathcal{O}(P), -\mathcal{O}(Q)), n) = \frac{3}{2}n^2 + \frac{5}{2}n + 1,$$

$$i(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)), n) = i(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)), n) = 2n^2 + 2n + 1.$$

2. WHEN ARE THREE POLYTOPES SMOOTH?

In this section, we consider the characterization problem of partially ordered sets yield smooth Fano polytopes.

First, we recall some definitions. Let $\mathcal{P} \subset \mathbb{R}^d$ be a Fano polytope.

- We call \mathcal{P} *centrally symmetric* if $\mathcal{P} = -\mathcal{P}$.
- We call \mathcal{P} *pseudo-symmetric* if there exists a facet \mathcal{F} of \mathcal{P} such that $-\mathcal{F}$ is also its facet. Note that every centrally symmetric polytope is pseudo-symmetric.
- A *del Pezzo polytope* of dimension $2k$ is a convex polytope

$$V_{2k} = \text{conv}(\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_{2k}, \pm(\mathbf{e}_1 + \dots + \mathbf{e}_{2k})).$$

Note that del Pezzo polytopes are centrally symmetric smooth Fano polytopes.

- A *pseudo del Pezzo polytope* of dimension $2k$ is a convex polytope

$$\tilde{V}_{2k} = \text{conv}(\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_{2k}, \mathbf{e}_1 + \dots + \mathbf{e}_{2k}).$$

Note that pseudo del Pezzo polytopes are pseudo-symmetric smooth Fano polytopes.

- Let us that \mathcal{P} *splits* into \mathcal{P}_1 and \mathcal{P}_2 if the convex hull of two Fano polytopes $\mathcal{P}_1 \subset \mathbb{R}^{d_1}$ and $\mathcal{P}_2 \subset \mathbb{R}^{d_2}$ with $d = d_1 + d_2$, i.e., by renumbering

$$\mathcal{P} = \text{conv}(\{(\alpha_1, 0), (0, \alpha_2) \in \mathbb{R}^d : \alpha_1 \in \mathcal{P}_1, \alpha_2 \in \mathcal{P}_2\}).$$

Then we write $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$.

There is well-known fact on the characterization of centrally symmetric or pseudo-symmetric smooth Fano polytopes.

- Any centrally symmetric smooth Fano polytope splits into copies of the closed interval $[-1, 1]$ or a del Pezzo polytope [16].
- Any pseudo-symmetric smooth Fano polytope splits into copies of the closed interval $[-1, 1]$ or a del Pezzo polytope or pseudo del Pezzo polytope [2, 16].

Let P and Q be partially ordered sets with $|P| = |Q| = d$. In this section, we consider when each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth.

First, we consider when $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth. For $1 \leq i \leq d$, we set $A_i(P) = \{I \in A(P) : |A| = i\}$.

Theorem 2.1. *For $d \geq 2$, let P and Q be partially ordered sets with $|P| = |Q| = d$. Then the following conditions are equivalent:*

- (i) $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is \mathbb{Q} -factorial;
- (ii) $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth;
- (iii) $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ splits into copies of the closed interval $[-1, 1]$ or a del Pezzo 2-polytope or pseudo del Pezzo 2-polytope;
- (iv) For any $I_1, I_2 \in A_2(P)$ with $I_1 \neq I_2$, $I_1 \cap I_2 = \emptyset$ and for any $J_1, J_2 \in A_2(Q)$ with $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$, and for any $I \in A_2(P)$ and for any $J \in A_2(Q)$, $|I \cap J| \neq 1$.

Proof. ((i) \Rightarrow (iv)) Let $p_{i_1} \prec p_{i_2} \prec \dots \prec p_{i_s}$ be a maximal chain of P . Then $x_{i_1} + x_{i_2} + \dots + x_{i_s} = 1$ is a facet of $\mathcal{C}(P)$, in particular, this is a facet of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$. Since $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is simplicial, this facet is a $(d-1)$ -simplex. Hence there exist just $d-s$ antichains $I_1, \dots, I_{d-s} \in A(P) \setminus A_1(P)$ such that for each I_k , $|\{p_{i_1}, p_{i_2}, \dots, p_{i_s}\} \cap I_k| = 1$. Since for each $j \in [d] \setminus \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$, there exists $i \in \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$ such that $\{i, j\}$ is an antichain of P , for each $j \in [d] \setminus \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$, there exists just one $i \in \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$ such that $\{i, j\}$ is an antichain of P . Then for $k \geq 3$, $A_k(P) = \emptyset$.

First, we assume that there exist $I_1, I_2 \in A_2(P)$ with $I_1 \neq I_2$ such that $I_1 \cap I_2 \neq \emptyset$. Let $I_1 = \{p_{i_1}, p_{i_2}\}$ and $I_2 = \{p_{i_1}, p_{i_3}\}$. Then we know that $\{p_{i_2}, p_{i_3}\}$ is not an antichain of P . Indeed, if $\{p_{i_2}, p_{i_3}\}$ is an antichain of P , then $\{p_{i_1}, p_{i_2}, p_{i_3}\}$ is also an antichain of P . Hence there exists a maximal chain $p_{j_1} \prec p_{j_2} \prec \dots \prec p_{j_t}$ of P such that $\{p_{i_2}, p_{i_3}\} \subset \{p_{j_1}, p_{j_2}, \dots, p_{j_t}\}$. Then since $\{p_{i_1}, p_{i_2}\}$ and $\{p_{i_1}, p_{i_3}\}$ are antichains of P , a facet $x_{j_1} + x_{j_2} + \dots + x_{j_t} = 1$ of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is not a $(d-1)$ -simplex.

Next, we assume that for any $I_1, I_2 \in A_2(P)$ with $I_1 \neq I_2$, $I_1 \cap I_2 = \emptyset$, and for any $J_1, J_2 \in A_2(Q)$ with $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$, and there exist $I \in A_2(P)$ and $J \in A_2(Q)$ such that $|I \cap J| = 1$. We let $I = \{p_{i_1}, p_{i_2}\}$ and $J = \{q_{i_1}, q_{i_3}\}$. Then $x_{i_2} - x_{i_3} = 1$ is a face of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ and this face is not simplex. Indeed, we set $\mathcal{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_{i_2} - x_{i_3} = 1\}$ and $\mathcal{H}^+ = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_{i_2} - x_{i_3} \leq 1\}$. Then every vertex of $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ belongs to \mathcal{H}^+ , and $\rho(\{p_{i_1}, p_{i_2}\}), \rho(\{p_{i_2}\}), -\rho(\{q_{i_1}, q_{i_3}\}), -\rho(\{q_{i_3}\})$ belong to \mathcal{H} . Since $(\rho(\{p_{i_1}, p_{i_2}\}) - (-\rho(\{q_{i_3}\}))) = (\rho(\{p_{i_2}\}) - (-\rho(\{q_{i_3}\}))) - (-\rho(\{q_{i_1}, q_{i_3}\}) - (-\rho(\{q_{i_3}\})))$, this face is not a simplex.

((iv) \Rightarrow (iii)) We assume that

$$A_2(P) = \{\{p_1, p_2\}, \dots, \{p_{2k-1}, p_{2k}\}, \{p_{2k+1}, p_{2k+2}\}, \dots, \{p_{2k+2l-1}, p_{2k+2l}\}\},$$

$$A_2(Q) = \{\{q_1, q_2\}, \dots, \{q_{2k-1}, q_{2k}\}, \{q_{2k+2l+1}, q_{2k+2l+2}\}, \dots, \{q_{2k+2l+2m-1}, q_{2k+2l+2m}\}\},$$

where k, l and m are nonnegative integers with $2k + 2l + 2m \leq d$. Then we have $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) = \text{conv}(\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_d, \pm(\mathbf{e}_1 + \mathbf{e}_2), \dots, \pm(\mathbf{e}_{2k-1} + \mathbf{e}_{2k}), \mathbf{e}_{2k+1} + \mathbf{e}_{2k+2}, \dots, \mathbf{e}_{2k+2l-1} + \mathbf{e}_{2k+2l}, -(\mathbf{e}_{2k+2l-1} + \mathbf{e}_{2k+2l+2}), \dots, -(\mathbf{e}_{2k+2l+2m-1} + \mathbf{e}_{2k+2l+2m}))$.

Hence $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ splits into copies of the closed interval $[-1, 1]$ or a del Pezzo 2-polytope or pseudo del Pezzo 2-polytope.

((iii) \Rightarrow (ii) \Rightarrow (i)) Since $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ splits into copies of the closed interval $[-1, 1]$ or a del Pezzo 2-polytope or pseudo del Pezzo 2-polytope, $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth. Moreover, in general, any smooth Fano polytope is simplicial. \square

Next, we consider when $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is smooth.

Theorem 2.2. *For $d \geq 2$, let P and Q be partially ordered sets with $|P| = |Q| = d$. Then the following conditions are equivalent:*

- (i) $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is \mathbb{Q} -factorial;
- (ii) $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is smooth;
- (iii) $I(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$ or
 $I(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$, and
 $A(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \dots, \{q_{i_d}\}\}$ or
 $A(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \dots, \{q_{i_d}\}, \{q_{i_1}, q_{i_2}\}\};$

Proof. ((i) \Rightarrow (iii)) We may assume that p_{i_1} is a minimal element of P and $I'(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\} \subset I(P)$. Then $x_{i_1} = 1$ is a facet of $\mathcal{O}(P)$, in particular, this is a facet of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$. Since $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is simplicial, this facet is a $(d-1)$ -simplex. Hence there is no poset ideal $I \in I(P)$ such that $p_{i_1} \in I$ and $I \notin I'(P)$. If there exists $I \in I(P)$ such that $p_{i_1} \notin I$, there exists a minimal element $p_{i_j} \in I$ of P . Then since $\{\{p_{i_1}, p_{i_j}\}\}$ is a poset ideal of P , we have $j = 2$. Hence we know that $I(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$ or $I(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$. Also, by the proof of Theorem 2.1, we may assume that for any $J_1, J_2 \in A_2(Q)$ with $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$.

We assume that $I(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$. If $\{q_{i_j}, q_{i_k}\}$ is an antichain of Q with $2 \leq j < k$, then $x_{i_1} - x_{i_k} = 1$ is a face of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and this face is not a simplex. Indeed, we set $\mathcal{H}_1 = \{(x_1, \dots, x_d) \in \mathbb{R}^d: x_{i_1} - x_{i_k} = 1\}$ and $\mathcal{H}_1^+ = \{(x_1, \dots, x_d) \in \mathbb{R}^d: x_{i_1} - x_{i_k} \leq 1\}$. Then every vertex of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ belongs to \mathcal{H}_1^+ . Also, $\rho(\{p_{i_1}\}), \rho(\{p_{i_1}, p_{i_2}\}), \dots, \rho(\{p_{i_1}, \dots, p_{i_{k-1}}\}), -\rho(\{q_{i_k}\})$ and $-\rho(\{q_{i_j}, q_{i_k}\})$ belong to \mathcal{H}_1 . Since $(-\rho(\{q_{i_k}\}) - \rho(\{p_{i_1}\})) = (-\rho(\{q_{i_j}, q_{i_k}\}) - \rho(\{p_{i_1}\})) + (\rho(\{p_{i_1}, \dots, p_{i_j}\}) - \rho(\{p_{i_1}\})) - (\rho(\{p_{i_1}, \dots, p_{i_{j-1}}\}) - \rho(\{p_{i_1}\}))$, this face is not a simplex. If $\{q_{i_1}, q_{i_j}\}$ is an antichain of Q with $3 \leq j$, then $-x_{i_1} + 2x_{i_2} = 1$ is a face of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and this face is not a simplex. Indeed, we set $\mathcal{H}_2 = \{(x_1, \dots, x_d) \in \mathbb{R}^d: -x_{i_1} + 2x_{i_2} = 1\}$ and $\mathcal{H}_2^+ = \{(x_1, \dots, x_d) \in \mathbb{R}^d: -x_{i_1} + 2x_{i_2} \leq 1\}$. Then each vertex of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ belongs to \mathcal{H}_2^+ . Also, $\rho(\{p_{i_1}, p_{i_2}\}), \dots, \rho(\{p_{i_1}, \dots, p_{i_d}\}), -\rho(\{q_{i_1}\})$ and $-\rho(\{q_{i_1}, q_{i_j}\})$ belong to \mathcal{H}_2 . Hence since $-x_{i_1} + 2x_{i_2} = 1$ has $d+1$ vertices, this face is not a simplex.

We assume that $I(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$. If $\{q_{i_j}, q_{i_k}\}$ is an antichain of Q with $2 \leq j < k$, then similarly, $x_{i_1} - x_{i_k} = 1$ is a face of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and this face is not a simplex. If $\{q_{i_1}, q_{i_j}\}$ is an antichain of Q with

$3 \leq j$, then $x_{i_2} - x_{i_j} = 1$ is a face of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and this face is not a simplex. Indeed, we set $\mathcal{H}_3 = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_{i_2} - x_{i_j} = 1\}$ and $\mathcal{H}_3^+ = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_{i_2} - x_{i_j} \leq 1\}$. Then every vertex of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ belongs to \mathcal{H}_3^+ , and $\rho(\{p_{i_2}\}), \rho(\{p_{i_1}, p_{i_2}\}), \dots, \rho(\{p_{i_1}, \dots, p_{i_{j-1}}\}), -\rho(\{q_{i_j}\}), -\rho(\{q_{i_1}, q_{i_j}\})$ belong to \mathcal{H}_3 . Since $(\rho(\{p_{i_2}\}) - \rho(\{p_{i_1}, p_{i_2}, p_{i_3}\})) = (\rho(\{p_{i_1}, p_{i_2}\}) - \rho(\{p_{i_1}, p_{i_2}, p_{i_3}\})) - (-\rho(\{q_{i_j}\}) - \rho(\{p_{i_1}, p_{i_2}, p_{i_3}\})) + (-\rho(\{q_{i_1}, q_{i_j}\}) - \rho(\{p_{i_1}, p_{i_2}, p_{i_3}\}))$, this face is not a simplex.

((iii) \Rightarrow (ii)) If $\mathcal{P} \subset \mathbb{R}^d$ is a smooth Fano polytope of dimension d , $\mathcal{P}' = \text{conv}(\mathcal{P}, \mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_{d+1}, -\mathbf{e}_{d+1}) \subset \mathbb{R}^{d+1}$ is also smooth. Also, if $d = 2$, then $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is smooth. Hence for $d \geq 2$, we know that $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is smooth.

((ii) \Rightarrow (i)) In general, any smooth Fano polytope is simplicial. \square

Finally, we consider when $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is smooth.

Theorem 2.3. *For $d \geq 2$, let P and Q be partially ordered sets with $|P| = |Q| = d$. Assume that P and Q have a common linear extension. Then the following conditions are equivalent:*

- (i) $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is \mathbb{Q} -factorial;
- (ii) $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is smooth;
- (iii) $I(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$ or $I(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$, and $I(Q) = \{\{q_{i_1}\}, \{q_{i_1}, q_{i_2}\}, \dots, \{q_{i_1}, \dots, q_{i_d}\}\}$ or $I(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \{q_{i_1}, q_{i_2}\}, \dots, \{q_{i_1}, \dots, q_{i_d}\}\}$.

Proof. ((i) \Rightarrow (iii)) By the proof of Theorem 2.2, We have $I(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$ or $I(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \dots, \{p_{i_1}, \dots, p_{i_d}\}\}$. Also, we have $I(Q) = \{\{q_{j_1}\}, \{q_{j_1}, q_{j_2}\}, \dots, \{q_{j_1}, \dots, q_{j_d}\}\}$ or $I(Q) = \{\{q_{j_1}\}, \{q_{j_2}\}, \{q_{j_1}, q_{j_2}\}, \dots, \{q_{j_1}, \dots, q_{j_d}\}\}$. Since P and Q have a common linear extension, we may assume that for any $1 \leq k \leq d$, $i_k = j_k$.

((iii) \Rightarrow (ii)) If $\mathcal{P} \subset \mathbb{R}^d$ is a smooth Fano polytope of dimension d , $\mathcal{P}' = \text{conv}(\mathcal{P}, \pm(\mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_{d+1})) \subset \mathbb{R}^{d+1}$ is also smooth. Also, if $d = 2$, then $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is smooth. Hence for $d \geq 2$, we know that $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is smooth.

((ii) \Rightarrow (i)) In general, any smooth Fano polytope is simplicial. \square

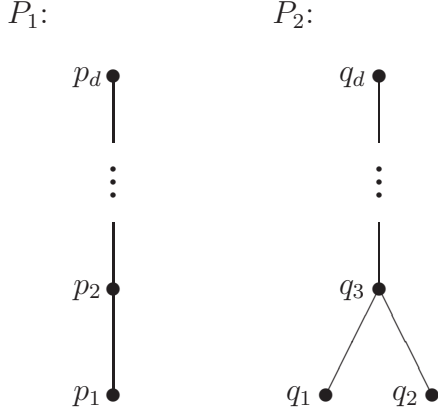
3. UNIMODULARLY EQUIVALENCE AND VOLUME

Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathbb{Z}^{d \times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral convex polytopes \mathcal{P} and \mathcal{Q} in \mathbb{R}^d of dimension d , we say that \mathcal{P} and \mathcal{Q} are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector w , such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in \mathbb{R}^d defined by U , i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all

$\mathbf{v} \in \mathbb{R}^d$. Clearly, if \mathcal{P} and \mathcal{Q} are unimodularly equivalent, then $i(\mathcal{P}, n) = i(\mathcal{Q}, n)$. Moreover, if \mathcal{P} is Fano, then $w = 0$.

Let P and Q be partially ordered sets with $|P| = |Q| = d$. We consider whether $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent when these polytopes are smooth. When $d = 2$ these polytopes are clearly unimodularly equivalent.

For $d \geq 3$ let P_1, P_2 be partially ordered sets as follows.



Each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth if and only if $P, Q \in \{P_1, P_2\}$.

Theorem 3.1. *For $d \geq 3$, let P and Q be partially ordered sets with $|P| = |Q| = d$. Assume that each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth. Then $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent. However, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not unimodularly equivalent to these polytopes. Moreover, if $P \neq Q$, then $\Gamma(\mathcal{O}(Q), -\mathcal{C}(P))$ is also smooth and is not unimodularly equivalent to $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$.*

Proof. Recall each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is smooth if and only if $P, Q \in \{P_1, P_2\}$. Hence we should consider the following 4 cases.

(The case $P = Q = P_1$) $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent, in particular, these polytopes are centrally symmetric. However, since $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not centrally symmetric, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not unimodularly equivalent to these polytopes.

(The case $P = Q = P_2$) Similarly, $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent, and $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not unimodularly equivalent to these polytopes.

(The case $P = P_1$ and $Q = P_2$) $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ are unimodularly equivalent, in particular, these polytopes are pseudo-symmetric. However, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not unimodularly equivalent to these polytopes, since $|\{v \in$

$V(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))): -v \in V(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))) \neq |\{v \in V(\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))): -v \in V(\Gamma(\mathcal{O}(P), -\mathcal{O}(Q)))\}|$, where we write $V(\mathcal{P})$ for the vertex set of a polytope \mathcal{P} .

(The case $P = P_2$ and $Q = P_1$) Similarly, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is not unimodularly equivalent to $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$. Moreover, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{O}(Q), -\mathcal{C}(P))$ are not unimodularly equivalent. Indeed, we assume that these polytopes are unimodularly equivalent. Then there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ such that $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)) = f_U(\Gamma(\mathcal{O}(Q), -\mathcal{C}(P)))$. Also for $v \in \{\pm \mathbf{e}_1, \pm(\mathbf{e}_1 + \mathbf{e}_2)\}$, there exists $v' \in \{\pm \mathbf{e}_1, \pm \mathbf{e}_2\}$ such that $f(v) = v'$.

If $f_U(\mathbf{e}_1) = \mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{e}_2$, we have

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f(-\mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2 \notin V(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)))$.

If $f_U(\mathbf{e}_1) = \mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = -\mathbf{e}_2$, we have

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (u_{31}, u_{32} - 1, u_{33}, \dots, u_{3d})$ and $f(-\mathbf{e}_3) = (-u_{31}, \dots, -u_{3d})$. Since $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is a $(-1, 0, 1)$ -polytope, $u_{32} = 0$ or 1 . Then $f(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = -\mathbf{e}_2$ or $f(-\mathbf{e}_3) = -\mathbf{e}_2$, contradiction.

If $f_U(\mathbf{e}_1) = -\mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{e}_2$, we have

$$U = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (u_{31}, u_{32} + 1, u_{33}, \dots, u_{3d})$ and $f(-\mathbf{e}_3) = (-u_{31}, \dots, -u_{3d})$. Since $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is a $(-1, 0, 1)$ -polytope, $u_{32} = 0$ or -1 . Then $f(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \mathbf{e}_2$ or $f(-\mathbf{e}_3) = \mathbf{e}_2$, contradiction.

If $f_U(\mathbf{e}_1) = -\mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = -\mathbf{e}_2$, we have

$$U = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f(-\mathbf{e}_2) = -\mathbf{e}_1 + \mathbf{e}_2 \notin V(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q)))$.

Therefore, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{O}(Q), -\mathcal{C}(P))$ are not unimodularly equivalent.

□

By Theorems 1.1 and 3.1, the following corollary immediately follows.

Corollary 3.2. *For any $d \geq 3$, there exist smooth Fano polytopes \mathcal{P} and \mathcal{Q} such that the following conditions satisfied:*

- \mathcal{P} and \mathcal{Q} have same Ehrhart polynomial.
- \mathcal{P} and \mathcal{Q} are not unimodularly equivalent.

Let \mathcal{P} be an integral convex polytope of dimension d . We write $\text{Vol}(\mathcal{P})$ for the *normalized volume* of \mathcal{P} ; it is equal to $d!$ times the usual Euclidean volume. It is known that if \mathcal{P}_1 is a d_1 -dimensional Gorenstein Fano polytope in \mathbb{R}^{d_1} and \mathcal{P}_2 is a d_2 -dimensional integral convex polytope in \mathbb{R}^{d_2} with $\mathbf{0} \in \mathcal{P}_1 \setminus \partial\mathcal{P}_1$, then

$$\text{Vol}(\mathcal{P}_1 \oplus \mathcal{P}_2) = \text{Vol}(\mathcal{P}_1)\text{Vol}(\mathcal{P}_2)$$

(see [1]). We let l, m, n be nonnegative integers and

$$\mathcal{P} = (\oplus_l L) \oplus (\oplus_m \tilde{V}_2) \oplus (\oplus_n V_2),$$

where L is the closed interval $[-1, 1]$. Since $\text{Vol}(L) = 2$, $\text{Vol}(\tilde{V}_2) = 5$ and $\text{Vol}(V_2) = 6$, we have $\text{Vol}(\mathcal{P}) = 2^l \cdot 5^m \cdot 6^n$.

Finally, we consider the volume of each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ when these polytopes are smooth.

Example 3.3. (i) Let $P = Q = P_1$. Then $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is unimodularly equivalent to $\oplus_d L$. Hence we know the normalized volume of each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is equal to 2^d by Theorem 1.1.

(ii) Let $P = Q = P_2$. Then $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is unimodularly equivalent to $(\oplus_{d-2} L) \oplus V_2$. Hence the normalized volume of each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is equal to $2^{d-2} \cdot 6$.

(iii) Let $P = P_1$ and $Q = P_2$. Then $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is unimodularly equivalent to $(\oplus_{d-2} L) \oplus \tilde{V}_2$. Hence the normalized volume of each of $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$, $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ and $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is equal to $2^{d-2} \cdot 5$. In particular, the normalized volume of $\Gamma(\mathcal{O}(Q), -\mathcal{O}(P))$ is also $2^{d-2} \cdot 5$.

REFERENCES

- [1] B. Braun, An Ehrhart Series Formula For Reflexive Polytopes, *Electron. J. Combin.* **13**(2006).
- [2] G. Ewald, On the classification of toric Fano varieties, *Disc. Comput. Geom.* **3** (1988), 49–54.
- [3] E. Ehrhart, “Polynomêes Arithmétiques et Méthode des Polyédres en Combinatoire”, Birkhäuser, Boston/Basel/Stuttgart, 1977.
- [4] H. Herzog and T. Hibi, “Monomial Ideals”, Graduate Text in Mathematics, Springer, 2011.
- [5] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws, in “Commutative Algebra and Combinatorics” (M. Nagata and H. Matsumura, Eds.), Advanced Studies in Pure Math., Volume 11, North-Holland, Amsterdam, 1987, pp. 93 – 109.
- [6] T. Hibi, Ed., “Gröbner Bases: Statistics and Software Systems,” Springer, 2013.
- [7] T. Hibi and N. Li, Chain polytopes and algebras with straitening laws, *Acta Math. Viet.*, to appear.
- [8] T. Hibi and N. Li, Unimodular equivalence of order and chain polytopes, *Math. Scand.*, to appear.
- [9] T. Hibi and K. Matsuda, Quadratic Gröbner bases of twinned order polytopes, arXiv:1505.04289.
- [10] T. Hibi, K. Matsuda, H. Ohsugi and K. Shibata, Centrally symmetric configurations of order polytopes, *J. Algebra*, to appear.
- [11] T. Hibi, K. Matsuda and A. Tsuchiya, Quadratic Gröbner bases arising from partially ordered sets, arXiv:1506.00802.
- [12] H. Ohsugi and T. Hibi, Quadratic initial ideals of root systems, *Proc. Amer. Math. Soc.* **130** (2002), 1913–1922.
- [13] H. Ohsugi and T. Hibi, Centrally symmetric configurations of integer matrices, *Nagoya Math. J.* **216** (2014), 153–170.
- [14] H. Ohsugi and T. Hibi, Reverse lexicographic squarefree initial ideals and Gorenstein Fano polytopes, arXiv:1410.4786.
- [15] R. P. Stanley, Two poset polytopes, *Disc. Comput. Geom.* **1** (1986), 9–23.
- [16] V. E. Voskresenskii and A. A. Klyachko, Toroidal Fano varieties and root system, *Math. USSR Izvestiya* **24**(1985), 221–244.

(Takayuki Hibi) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: `hibi@math.sci.osaka-u.ac.jp`

(Kazunori Matsuda) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: `kaz-matsuda@math.sci.osaka-u.ac.jp`

(Akiyoshi Tsuchiya) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: `a-tsuchiya@cr.math.sci.osaka-u.ac.jp`